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On the viscous flow in the nose region of a symmetric blunt body in hypersonic flow

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The flow field in the nose region of a blunt body in hypersonic flow is studied by considering the transport of vorticity and enthalpy. The entire region between body and shock is considered to be viscous, not necessarily thin in comparison with the nose radius of the body and to be of slowly varying density. The (given) post-shock vorticity need not be small and the density ratio ρ_∞/ρ_s may either be small or near unity, the analysis being valid asymptotically at both limits.

It is foundthat the vorticity equation may be uncoupled from the total enthalpy equation if $\mu\sqrt{\rho}$ is constant. While the equations are not expected to be necessarily restricted to the immediate vicinity of the stagnation line, only there can the solution be written down explicitly; elsewhere, numerical integration is required.

1. Introduction

The problem of the interaction of the external flow field with the flow in the boundary layer of blunt bodies in hypersonic flow has been analysed (Hayes & Probstein 1959; Cheng 1961, 1963) as an extension of boundary-layer theory; the boundary-layer solution is matched in a suitable manner to the non-uniform flow produced by the curved bow shock wave. Bush (1964) has shown that, for consistency, one requires to divide the region between body and shock into several layers, adjacent layers requiring individual matching.

The object of the following analysis is to obtain a single, approximate representation of the flow field valid everywhere between body and shock, thus obviating internal matching problems. This is done by considering the distribution of vorticity in the flow, which is to be considered viscous everywhere. The advantages are, first, there is no explicit pressure dependence in the vorticity equation, secondly, the post-shock vorticity distribution provides a simple boundary condition not requiring the simultaneous solution of two equations, which is required if the problem is tackled in terms of momentum distributions. The disadvantage is that the equation governing the distribution of vorticity cannot be expressed solely in terms of vorticity, compressibility effects aside; the velocity components appear explicitly in the inertia terms. This difficulty is circumvented by applying a transformation which is essentially the well-known von Mises transformation, which removes the offending terms, but recreates the difficulty in the viscous terms. We then utilize a device constructed by Lighthill (1950) specifically to deal with this latter problem. An unforseen advantage of

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this manoeuvre is that the coupling between the vorticity and enthalpy fields occurs through the combination $\mu\sqrt{\rho}$; since, for constant pressure and ignoring non-equilibrium effects, this is constant to within 5 % for air between 500 °K and 12,000 °K (see, for example, Hansen 1959) and, for a perfect gas, is dependent only on the square root of the pressure, choosing a suitable mean value uncouples the vorticity and enthalpy equations.

Since the region over which the analysis is applied is not necessarily a thin layer, the equations are formulated in terms of an orthogonal co-ordinate system having the body and the shock as members of one of its families. As a result we may write down the solution only in the immediate neighbourhood of the stagnation line; elsewhere in the region of validity of the assumptions, we require to solve the equations numerically.

2. The vorticity equation

The equations governing the motion of a steady compressible fluid may be written as

$$\nabla.(\rho\mathbf{v}) = 0,$$

$$\begin{aligned}-\rho\mathbf{v}\times\boldsymbol{\omega}+\tfrac{1}{2}\rho\nabla v^2 = {} & -\nabla p+\tfrac{4}{3}\nabla(\mu\nabla.\mathbf{v})+\nabla(\mathbf{v}.\nabla\mu)\\ & -\mathbf{v}\nabla^2\mu+\nabla\mu\times\boldsymbol{\omega}-(\nabla.\mathbf{v})\nabla\mu\\ & -\nabla\times\nabla\times(\mu\mathbf{v}),\end{aligned}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{v}$, together with an equation governing enthalpy transport. Taking the curl of the equation we see that the corresponding equation for the vorticity is

$$\begin{aligned}-\nabla\times(\rho\mathbf{v}\times\boldsymbol{\omega})+\tfrac{1}{2}\nabla\rho\times\nabla\mathbf{v}^2 = {} & -\nabla\times(\mathbf{v}\nabla^2\mu)+\nabla\times(\nabla\mu\wedge\boldsymbol{\omega})\\ & -\nabla\times\{(\nabla.\mathbf{v})\nabla\mu\}\\ & -\nabla\times\nabla\times\nabla\times(\mu\mathbf{v}). \qquad (1)\end{aligned}$$

In a general orthogonal co-ordinate system with length element $ds^2 = \sum_i h_i^2\,dx_i^2$, and restricting the analysis to two-dimensional or axisymmetric flows so that $u_3 \equiv 0$, $\partial/\partial x_3 \equiv 0$, (1) becomes

$$\begin{aligned}&-\frac{\rho h_3}{h_1h_2}\left\{\frac{u_1}{h_1}\frac{\partial}{\partial x_1}\left(\frac{\omega_3}{h_3}\right)+\frac{u_2}{h_2}\frac{\partial}{\partial x_2}\left(\frac{\omega_3}{h_3}\right)\right\}\\ &\quad = -\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left\{\frac{h_2}{h_3h_1}\frac{\partial}{\partial x_1}\left(\frac{h_3^2}{h_1h_2}\frac{\mu\omega_3}{h_3}\right)\right\}+\frac{\partial}{\partial x_2}\left\{\frac{h_1}{h_2h_3}\frac{\partial}{\partial x_2}\left(\frac{h_3^2}{h_1h_2}\frac{\mu\omega_3}{h_3}\right)\right\}\right]\\ &\qquad -\{\nabla\times(\mathbf{v}\nabla^2\mu)\}_3+\{\nabla\times(\nabla\mu\times\boldsymbol{\omega})\}_3,\end{aligned}$$

where we note that $\omega_1 \equiv \omega_2 \equiv 0$ and where we have neglected derivatives of the density. If we define S by the equation

$$\begin{aligned}-\frac{\rho u_1}{h_1}\frac{h_3}{h_1h_2}S = {} & -\frac{1}{h_1h_2}\frac{\partial}{\partial x_1}\left\{\frac{h_2}{h_3h_1}\frac{\partial}{\partial x_1}\left(\frac{h_3^2}{h_1h_2}\frac{\mu\omega_3}{h_3}\right)\right\}\\ & -\frac{1}{h_1h_2}\frac{\partial}{\partial x_2}\left\{\frac{h_1\omega_3}{h_2h_3}\frac{\partial}{\partial x_2}\left(\frac{h_3^2\mu}{h_1h_2}\right)\right\}-\{\nabla\times(\mathbf{v}\nabla^2\mu)\}_3+\{\nabla\times(\nabla\mu\times\boldsymbol{\omega})\}_3,\end{aligned}$$

and let
$$\Omega = -\frac{\omega_3}{h_3} = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial x_2}(h_1 u_1) - \frac{\partial}{\partial x_1}(h_2 u_2)\right\},$$
then we have
$$\frac{\rho u_1}{h_1}\frac{\partial \Omega}{\partial x_1} + \frac{\rho u_2}{h_2}\frac{\partial \Omega}{\partial x_2} = \frac{1}{h_3}\frac{\partial}{\partial x_2}\left(\frac{\mu h_3}{h_2^2}\frac{\partial \Omega}{\partial x_2}\right) + \frac{\rho u_1}{h_1} S. \tag{2}$$

We define the stream function ψ by the equations
$$\frac{\partial \psi}{\partial x_1} = -\rho h_3 h_1 u_2, \quad \frac{\partial \psi}{\partial x_2} = \rho h_2 h_3 u_1,$$
with $\psi = 0$ on the body. We take $h_1 dx_1$ to be the element of distance along members of a one parameter family of curves of which the shock wave and the body are members. Let Ψ be the parameter where we define Ψ to be ψ/ψ_s where ψ_s is the value of ψ on the upstream side of the shock wave at a given value of x_1; ψ_s is then a function of x_1 only. Lines of constant Ψ are shown in figure 1. We take $h_2 dx_2$ to be the distance element along members of the orthogonal net.

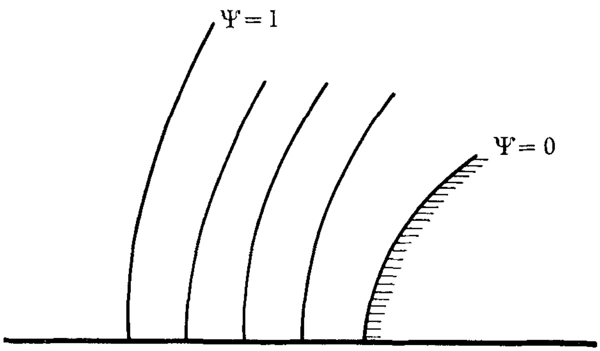

FIGURE 1. Lines of constant Ψ.

We transform from co-ordinates (x_1, x_2) to (x_1, Ψ) retaining velocities conjugate to x_1 and x_2. In this transformation $\partial/\partial x_1$ is replaced by
$$\frac{\partial}{\partial x_1} - \frac{\rho h_3 h_1 u_2}{\psi_s}\frac{\partial}{\partial \Psi} - \frac{\Psi}{\psi_s}\frac{d\psi_s}{dx_1}\frac{\partial}{\partial \Psi}$$
and $\partial/\partial x_2$ by $(\rho_2 h_3 u_1/\psi_s)\,\partial/\partial\Psi$. (2) then gives
$$\frac{\partial \Omega}{\partial x_1} - \frac{\Psi}{\psi_s}\frac{d\psi_s}{dx_1}\frac{\partial \Omega}{\partial \Psi} = \frac{h_1 h_2}{\psi_s}\frac{\partial}{\partial \Psi}\left(\frac{\rho h_3^2}{h_2}\frac{\mu u_1}{\psi_s}\frac{\partial \Omega}{\partial \Psi}\right) + S. \tag{3}$$

Near $\Psi = 0$,
$$\Omega \simeq \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x_2}(h_1 u_1)$$
$$= \frac{\rho u_1}{h_1 \psi_s}\frac{\partial}{\partial \Psi}(h_1 u_1),$$
from which, near $\Psi = 0$, we see that
$$u_1 \simeq \surd(2\psi_s\,\Omega(x_1, 0)\,\Psi/\rho).$$

Inserting this expression in the right-hand side of (3) we have

$$\frac{\partial \Omega}{\partial x_1} - \frac{\Psi}{\psi_s}\frac{d\psi_s}{dx_1}\frac{\partial \Omega}{\partial \Psi} = \frac{h_1 h_2}{\psi_s^{\frac{3}{2}}}\{2\Omega(x,0)\}^{\frac{1}{2}}\frac{\partial}{\partial \Psi}\left(\frac{\mu\rho^{\frac{1}{2}}h_3^2}{h_2}\Psi^{\frac{1}{2}}\frac{\partial \Omega}{\partial \Psi}\right) + S. \tag{4}$$

This is essentially Lighthill's (1950) device for removing the awkward u_1 produced by the von Mises transformation. He found that the approximation was adequate in favourable pressure gradients. He pointed out that the approximation is good near the body surface, but cannot really be expected to hold elsewhere, but that, where the approximation is bad, the whole term in which it is contained is small, so that the resulting error may be expected to be small also. In our case, we would hope that the approximation was adequate if $\partial(\Psi^{\frac{1}{2}}\,\partial\Omega/\partial\Psi)/\partial\Psi$ was small away from $\Psi = 0$.

Now $\mu\sqrt{\rho}$ is constant to within 5 % for equilibrium air between 500 °K and 12,000 °K at constant pressure, and varies only as the square root of the pressure for a perfect gas. Thus if we replace $\mu\sqrt{\rho}$ by a suitable mean value, we may uncouple the vorticity equation from the enthalpy equation.

The enthalpy equation may be treated in an exactly similar manner provided the Prandtl number is unity, so that terms involving squares of velocities do not appear. Total enthalpy and vorticity obey the same field equations, but of course satisfy different boundary conditions. Lighthill pointed out that when this device is applied to the enthalpy equation (he actually applied it to the temperature equation) it holds asymptotically for large values of the Prandtl number. He found, however, that the agreement was still good at a Prandtl number of unity.

3. Orders of magnitude

We introduce the ratio $\epsilon = \rho_\infty/\rho_s = (\gamma-1)/(\gamma+1)$, postulate that ϵ is small, and that $\partial\Omega/\partial x_1 = O(\epsilon\,\partial\Omega/\partial x_2)$. Since we have assumed that density variations between body and shock-wave are negligibly small, which only requires that $\partial\rho/\partial x_1 = O(\epsilon\rho_s/L)$, where L is a typical dimension in the x_1 direction, we take variations in μ to be negligible in the same sense; this seems plausible on physical grounds. Then, considering the magnitude of terms in $(\rho u_1/h_1)\,S$, we see that they are at most of order ϵ times the first term on the right-hand side of (2) and therefore are to be neglected in comparison with it.

We note that the result of applying this order-of-magnitude analysis is an equation identical to that which would have been produced by a boundary-layer analysis, which would be valid in rather different circumstances; the same terms would have been discarded, but for a different reason. However, if instead of putting $\partial/\partial x_1 = O(\epsilon\,\partial/\partial x_2)$ we put $\partial/\partial x_1 = O(\epsilon\,\delta^c\,\partial/\partial x_2)$, where δ is the dimensionless thickness of the boundary layer and c is some exponent greater than zero, then the vorticity equation already derived holds either for $\epsilon \ll 1$ with $\delta = O(1)$ or $\delta \ll 1$ with $\epsilon = O(1)$.

Thus, provided the boundary conditions are applied in forms valid both for $\epsilon \ll 1$ and $\epsilon = O(1)$ which is equivalent to $\delta \ll 1$, the approximations used are valid at both ends of the range $0 < \epsilon < 1$ for a given shock shape. To find Ω as a function of ϵ for a given body, one would require to extract from the above family of solutions for $R_b < R_s < \infty$ those compatible with some relation between R_s, R_b and ϵ; R_b and R_s denote the radii of curvature of the body and the shock wave, on the centre line.

4. Solution in series

If we concern ourselves with the solution in the vicinity of the stagnation point, then, replacing $\mu\sqrt{\rho}$ by a suitable mean value and assuming that h_1, h_2 and h_3 are functions of x_1 only, and if we, further, define X by

$$X = \sqrt{2}\,\mu\rho^{\frac{1}{2}} \int_0^{x_1} \frac{h_1 h_3^{\frac{2}{3}}}{\psi_s^{\frac{4}{3}}} \Omega^{\frac{1}{2}}(x_1, 0)\, dx_1,$$

we have

$$\frac{\partial \Omega}{\partial X} - \frac{\Psi}{\psi_s}\frac{d\psi_s}{dX}\frac{\partial \Omega}{\partial \Psi} = \frac{\partial}{\partial \Psi}\left(\Psi^{\frac{1}{2}}\frac{\partial \Omega}{\partial \Psi}\right), \tag{5}$$

which equation is linear in Ω. If we let $\Omega = \omega X$, then

$$\omega - \Psi \frac{d\omega}{d\Psi} = \frac{d}{d\Psi}\left(\Psi^{\frac{1}{2}}\frac{d\omega}{d\Psi}\right),$$

assuming $\psi_s = \psi_0 X + \ldots$ to first order in X. We may now define $\omega^* = \omega/\omega(0)$, for our convenience. Then

$$\omega^* = \exp\left(-\tfrac{2}{3}\Psi^{\frac{3}{2}}\right)\left\{A\Psi^{\frac{1}{2}}\,{}_1F_1\left(\tfrac{5}{3}, \tfrac{4}{3}, \tfrac{2}{3}\Psi^{\frac{3}{2}}\right) + B\,{}_1F_1\left(\tfrac{4}{3}, \tfrac{2}{3}, \tfrac{2}{3}\Psi^{\frac{3}{2}}\right)\right\},$$

where ${}_1F_1(a, b, z)$ is the confluent hypergeometric function. The boundary conditions to be applied are $\omega^*(1) = \alpha$, $\omega^*(0) = 1$; α is still to be determined and is obtained from the derivative of ω at $\Psi = 1$, as we show later. Since $B = 1$, A is easily obtained in terms of α.

From the u_1 momentum equation we have that

$$\frac{h_1}{h_2}\rho u_2^2 \frac{\partial}{\partial x_2}\left(\frac{u_1}{u_2}\right) = -\frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}(\mu\Omega).$$

Now $u_1/u_2 \simeq \tan\theta_s$, where θ_s is the deviation of the streamline from the line of symmetry immediately behind the shock wave, as in figure 2; near the line of symmetry, we would expect θ_s to vary little near the shock wave. Hence, near $x_1 = 0$ and $\Psi = 1$, we put

$$\frac{\partial p}{\partial x_1} \simeq \frac{\partial}{\partial x_2}(\mu\Omega).$$

Now

$$\partial p/\partial x_1 \simeq -2\rho_\infty U_\infty^2 x_1/R_s^2;$$

hence

$$\frac{\partial \Omega}{\partial x_2} \simeq -\frac{2\epsilon^{\frac{1}{2}}\Omega_s \Re_\infty}{(\epsilon-1)^2\{1+\gamma M_\infty^2(1-\epsilon)\}^{\frac{1}{2}} R_s},$$

where we have used the result that $\mu\sqrt{(\rho/p)}$ is invariant across the shock wave, where $\Re_\infty = U_\infty R_s/\nu_\infty$ and where $\Omega_s = U_\infty x_1(1-\epsilon)^2/\epsilon R_s^2$. Note that, in the region between the body and the shock wave we were prepared to ignore variations in $\sqrt{p}$ to uncouple the vorticity equation from the enthalpy equation. We cannot, of course, ignore the change in $\sqrt{p}$ as the fluid crosses the shock wave. Hence the outer boundary condition on ω^* is to be derived from

$$\frac{d\omega^*}{dx_2} = -\frac{2\epsilon^{\frac{1}{2}}\mathfrak{R}_\infty\alpha}{\{1+\gamma M_\infty^2(1-\epsilon)\}^{\frac{1}{2}}R_s},$$

where $\alpha = \omega(1)/(1-\epsilon)^2\omega(0)$; note that $\omega(1)/(1-\epsilon)^2$ is finite as $\epsilon \to 1$. Now $d/dx_2 = (h_3\rho u_1/\psi_s)\,d/d\Psi$, putting $h_2 = h_1 = 1$, and $\psi_s \simeq \rho h_3 u_2 x_1$ we have

$$\frac{d}{dx_2} \simeq \frac{1}{x_1}\frac{u_1}{u_2}\frac{d}{d\Psi} = \frac{\tan\theta_s}{x_1}\frac{d}{d\Psi}.$$

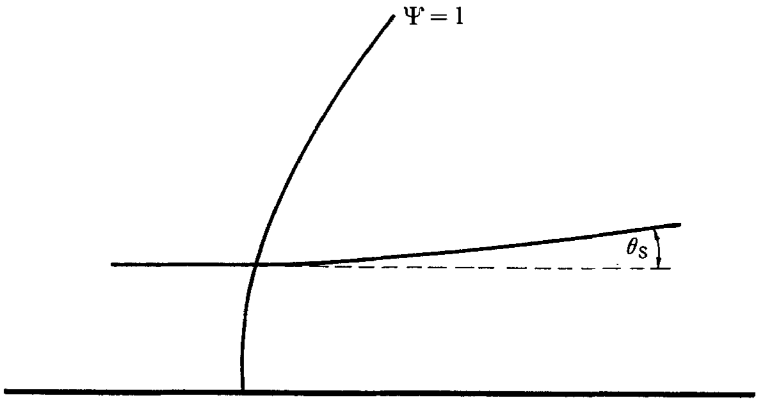

FIGURE 2. Streamline deviation near $\Psi = 1$ and $x_1 = 0$.

Now, $\tan\theta_s \simeq x_1(1-\epsilon)/\epsilon R_s$ for constant ϵ. Hence, at $\Psi = 1$,

$$\frac{d\omega^*}{d\Psi} = -\frac{2\mathfrak{R}_\infty\epsilon^{\frac{3}{2}}\alpha}{(1-\epsilon)\{1+\gamma M_\infty^2(1-\epsilon)\}^{\frac{1}{2}}};$$

this condition enables α to be determined uniquely. We find that

$$\omega^* = \exp\left(-\tfrac{2}{3}\Psi^{\frac{3}{2}}\right)\{(0{\cdot}8677\alpha - 1{\cdot}412)\Psi^{\frac{1}{2}}\,{}_1F_1(\tfrac{5}{3},\tfrac{4}{3},\tfrac{2}{3}\Psi^{\frac{3}{2}}) + {}_1F_1(\tfrac{4}{3},\tfrac{2}{3},\tfrac{2}{3}\Psi^{\frac{3}{2}})\},$$

where $\alpha = 1/(3{\cdot}048 + 8{\cdot}945\beta)$ and

$$\beta = \frac{\mathfrak{R}_\infty\epsilon^{\frac{3}{2}}}{(1-\epsilon)\{1+\gamma M_\infty^2(1-\epsilon)\}^{\frac{1}{2}}}.$$

We note that $\alpha\beta$ is bounded as $\beta \to \infty$, that is as $\epsilon \to 1$ so that $d\omega^*/d\Psi$ is bounded and decreases monotonically from zero to $-0{\cdot}2237$ as ϵ increases from zero to one. Figure 3 shows ω^* for various values of β; figure 4 shows $\sqrt{\Psi}\,d\omega^*/d\Psi$. We note in passing that, as β increases, the scale of x_2, that is distance in the Ψ direction, decreases. Near $\Psi = 1$, $d\omega^*/dx_2$ decreases as β increases.

We note that the curves for $\beta = 10$ and $\beta = 100$ are essentially coincident, and represent, on this simple approach, the conventional boundary-layer limit of the analysis. As Bush (1964) has pointed out, the limit $\beta = 0$ has no very great physical significance. However, we note that, as $\beta \to 0$, ω^* is bounded and non-zero for all Ψ. In particular, $\omega^*(1) = \omega(1)/\omega(0)$ is non-zero. Now, $\omega(1) \to 0$ as

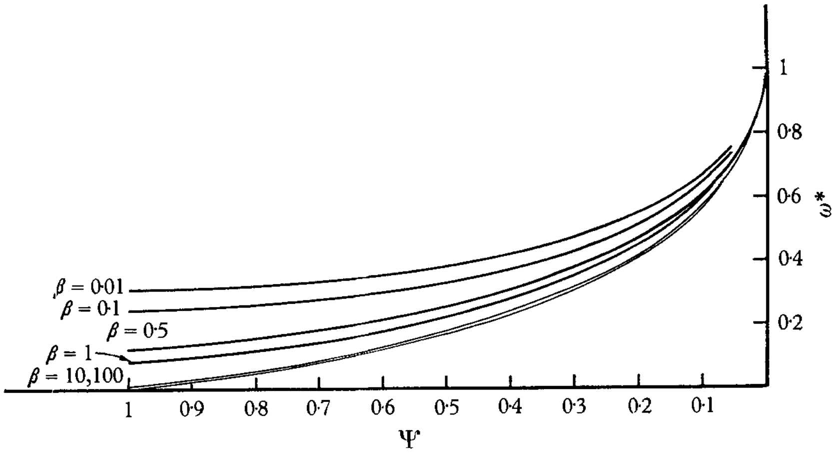

FIGURE 3. Variation of ω^* with Ψ'.

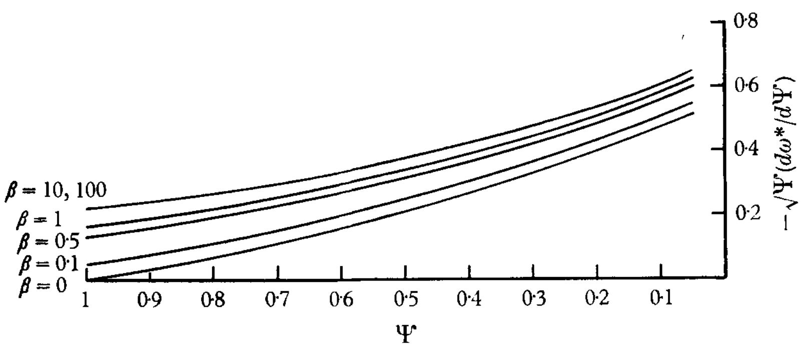

FIGURE 4. Variation of $-\sqrt{\Psi'}(d\omega^*/d\Psi')$ with Ψ'.

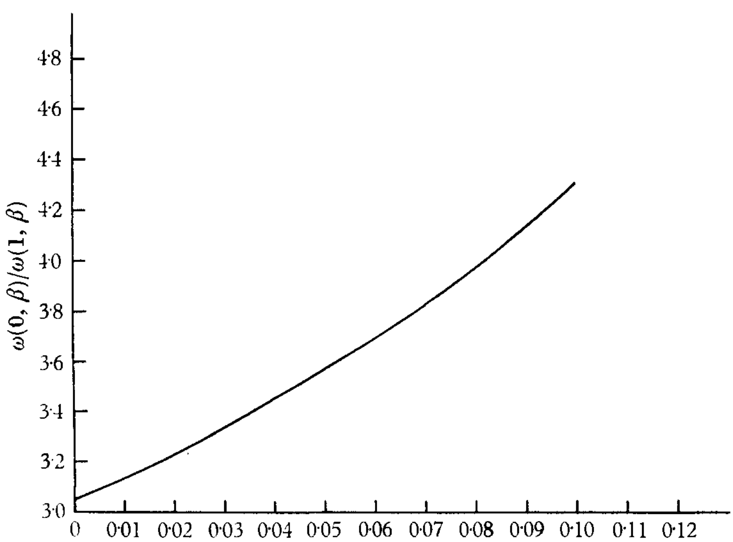

FIGURE 5. Variation of $\omega(0, \beta)/\omega(1, \beta)$ with β.

$\beta \to 0$, where we take this limit to imply the limit of low Reynolds numbers. Hence, we see that $\omega(0) \to 0$ as we would expect. Further

$$\omega(0,\beta)/\omega(1,\beta) = (0{\cdot}328 - 0{\cdot}963\beta)^{-1}$$

in an obvious notation. This gives the relation between the vorticity at the body surface and the vorticity at the outer edge of the viscous layer in low-Reynolds-numbers flow. This is shown in figure 5.

It is interesting to note that the parameter β is sandwiched, in order of magnitude, between the parameters $1/K$ and $1/D$ of Bush (1964), for putting $\delta \sim \epsilon^{-1} M_\infty^{-2}$, we have that $\beta \sim \epsilon^2 \mathfrak{R}_\infty \sqrt{\delta}$; the exponent of δ comes from the assumed temperature dependence of μ, the exponent of ϵ from insisting that layers characterized by parameters $1/K = \epsilon \mathfrak{R}_\infty \sqrt{\delta}$ and $1/D = \epsilon^{\frac{5}{2}} \mathfrak{R}_\infty \sqrt{\delta}$ should be considered as one. We recall that $K = O(1)$ characterized the viscous shock layer and $D = O(1)$ the viscous body layer.

5. Comparison with more exact analyses

Let us now compare the predictions of the present theory with those of Bush (1964). First, however, we note that we may only obtain an expression for the skin friction as a function of Ω_s and, likewise, the heat transfer as a function of H_s; the quantities Ω_s and H_s are the values of Ω and H respectively at the inner edge of the shock-structure region, in Bush's terminology, or at the outer edge of the shock layer of Cheng (1961). These quantities are not readily determinable for given values of ϵ, M_∞ and $\mathfrak{R}_\infty$.

We proceed to take the work of Bush as definitive for one set of values of ϵ, M_∞ and $\mathfrak{R}_\infty$, noting that the similarity parameter in that analysis,

$$K = \sqrt{2} M_\infty / \sqrt{\epsilon} \mathfrak{R}_\infty,$$

differs from the parameter $\beta = \epsilon^{\frac{5}{2}} \mathfrak{R}_\infty M_\infty^{-1}$ of the present analysis. We approximate the dependence of the skin-friction and heat-transfer coefficients on K to an inverse power of K and choose the dependence of Ω_s and H_s on ϵ, M_∞ and $\mathfrak{R}_\infty$ so that the predictions of the coefficients would be expected to agree approximately to within a constant factor. In this process, we ignore the dependence of α on β; we rely on this dependence to modify the power-law curve to fit more exactly Bush's results. We determine the constant for a particular set of values of ϵ, M_∞ and $\mathfrak{R}_\infty$. The reasonableness of the assumptions underlying the present analysis must be judged by observing how well the predictions of the skin-friction and heat-transfer coefficients for different groups of values of ϵ, M_∞ and $\mathfrak{R}_\infty$ agree with those of Bush.

First, we define Ω' to be the vorticity distribution behind a thin shock wave of radius R_b or R_s. Then $\Omega' = U_\infty x_1 / \epsilon R_s^2$ and we let $\Omega_s = C^*(\epsilon, M_\infty, \mathfrak{R}_\infty)\, \Omega'$, where the function C^* is to be determined. Secondly, we let H be the total enthalpy of the free stream and let $H_s = C(\epsilon, M_\infty, \mathfrak{R}_\infty)\, H_\infty$, where C is to be determined. The skin-friction coefficient C_f, defined by Bush, takes the value $M_\infty C^*/\alpha \sqrt{\epsilon}$ in terms of the present analysis and the heat-transfer coefficient C_h takes the value

$\epsilon^{\frac{3}{2}} M_\infty^3 H_\infty^{-1} R_s \,\partial H/\partial x_2|_{\Psi=0}$. For unit Prandtl number, H and Ω satisfy the same equation so that it is readily shown that

$$\left.\frac{\partial H}{\partial x_2}\right|_{\Psi=0} = H_\infty h_1 h_2 h_3 \left\{\frac{\exp\left(\frac{2}{3}\right) - B\,{}_1F_1\left(\frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right)}{{}_1F_1\left(\frac{5}{3}, \frac{4}{3}, \frac{2}{3}\right)}\right\} C^* M_\infty \epsilon^{-\frac{3}{2}} \alpha^{-1} R_s^{-1} \Re_\infty^{-1},$$

where B is the value of H at $\Psi = 0$. Hence

$$C_h = (0{\cdot}872 - 1{\cdot}415B)\, M_\infty^4\, CC^* \alpha^{-1} \Re_\infty^{-1}.$$

Let us now consider axisymmetric flows. From the graph of C_f against K given by Bush we deduce that C_f varies approximately like $1/\sqrt{K}$, that is $\epsilon^{\frac{1}{4}} \Re_\infty^{\frac{1}{2}} M^{-\frac{1}{2}}$. Thus C^* must be taken to be proportional to $\epsilon^{\frac{3}{4}} \Re_\infty^{\frac{1}{2}} M_\infty^{-\frac{3}{2}}$. Fitting the predictions of C_f for $\epsilon = 0{\cdot}05$, $K = 1{\cdot}5$, we find that

$$C^* = \tfrac{1}{8} \epsilon^{\frac{3}{4}} \Re_\infty^{\frac{1}{2}} M_\infty^{-\frac{3}{2}},$$

so that

$$\Omega_s = \tfrac{1}{8} \epsilon^{\frac{3}{4}} \Re_\infty^{\frac{1}{2}} M_\infty^{-\frac{3}{2}} \Omega'. \tag{6}$$

Figure 6 shows graphs of C_f determined by the present analysis expressed as a function of K. We also show Bush's results and the inverse-square-root-power curve. The agreement is seen to be quite satisfactory, with an error of less than 10 % in the range $1 \leqslant K \leqslant 7$.

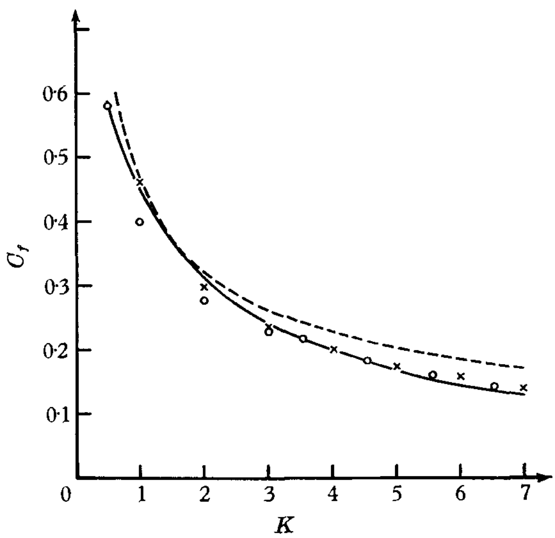

FIGURE 6. Variation of C_f with K. ×, $\epsilon = 0{\cdot}05$; ○, $\epsilon = 0{\cdot}01$; —, results of Bush (1964); – – –, ordinates proportional to $1/\sqrt{K}$.

We repeat the process for C_h. Bush's results correspond to $B = 0$ so that $C_h = 0{\cdot}872 M_\infty^4\, CC^*/\alpha\Re_\infty$, and correspond to a Prandtl number of $\frac{3}{4}$; our analysis requires that the Prandtl number is unity. In order to make any comparison at all, we are obliged to assume that Prandtl number plays a minor role provided it is of order unity. The effect is that our predictions concerning the quantity H_s will be slightly in error. From Bush's graph, we find that $C_h \sim K^{-\frac{3}{4}}$ so that $C \sim \Re_\infty^{\frac{5}{4}} \epsilon^{-\frac{3}{8}} M_\infty^{-\frac{13}{4}}$. Again, fitting at $\epsilon = 0{\cdot}05$, $K = 1{\cdot}5$, we find that

$$H_s = 0{\cdot}72 \Re_\infty^{\frac{5}{4}} \epsilon^{-\frac{3}{8}} M_\infty^{-\frac{13}{4}} H_\infty. \tag{7}$$

Figure 7 shows values of C_h determined by the present analysis expressed as a function of K. The agreement here is very satisfactory, the error being less than 1 % for $1{\cdot}5 \leqslant K \leqslant 7$. The degree of agreement leads us to infer that the expressions (6) and (7) for the vorticity and total enthalpy at the inner edge of the shock-structure region should give quite reasonable estimates of these quantities in the actual flow, quite apart from providing the appropriate outer boundary conditions for more or less accurate values of skin friction and heat transfer, subject to the remarks above concerning Prandtl number.

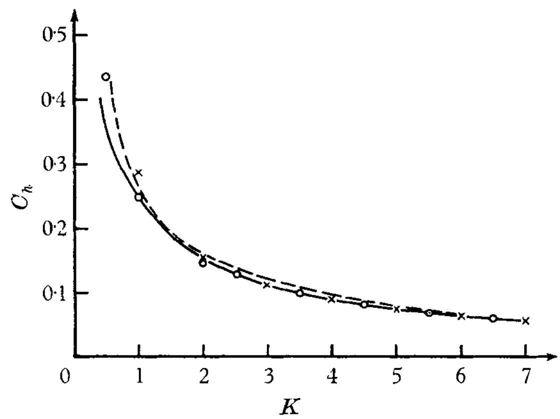

FIGURE 7. Variation of C_h with K. ×, $\epsilon = 0{\cdot}05$; ○, $\epsilon = 0{\cdot}01$; —, results of Bush (1964); – – –, ordinates proportional to $K^{-\frac{3}{4}}$.

It is interesting to note that expressions (6) and (7) may be written in the form

$$\Omega_s = \tfrac{1}{8}k^{\frac{1}{2}}(\epsilon M_\infty^2)^{-\frac{3}{4}}\, U x_1 R_s^{-2}$$

and

$$H_s = 0{\cdot}72 k^{\frac{5}{4}}(\epsilon M_\infty^2)^{-\frac{13}{8}} H_\infty,$$

where $k = \epsilon \mathfrak{R}_\infty$ is the parameter K^2 of Cheng (1961). From Cheng's results, we see that, on his model, ϵM_∞^2 is a function of k only, for given free-stream conditions. Thus Ω_s and H_s are determined in terms of k alone so that we may make a direct comparison with the values calculated by Cheng. Only one set of results reported by Cheng, those displayed in his figure 5.3 corresponding to a Prandtl number of $\frac{3}{4}$, are suitable for comparison; in these, the value of ϵ is $0{\cdot}25$, which is perhaps hardly sufficiently small to satisfy the condition $\epsilon \ll 1$. However, the values of Ω_s and H_s disagree so markedly that there can be no doubt that expressions (6) and (7) in no way represent the actual values of vorticity and total enthalpy at the inner edge of the shock-structure region. In particular, the present analysis assumes that Ω increases monotonically from the outer edge of the flow field inwards, which is not the case in the results presented by Cheng in figure 5.3.

We may conclude, however, in view of the good agreement over a fivefold range in ϵ with the results of Bush, an agreement between analyses with different dependence on ϵ, M_∞ and $\mathfrak{R}_\infty$, that the replacing of u_1 by $\sqrt{\{2\psi_s\, \Omega(x_1, 0)\, \Psi/\rho\}}$ in the von Mises transformation in some sense mirrors reality in flows with $\epsilon \ll 1$.

6. Solutions away from the stagnation point

It will be noted that we have chosen the co-ordinate system in such a way that the validity of the formulation of the equation is not restricted to the stagnation region. Further, we might expect that the arguments for discarding the various terms should hold outside the stagnation region. However, the metric elements h_1, h_2, h_3 will, in general, be functions of x_1 and x_2, determined by the particular configuration under consideration, so that (4), with the S term deleted from the right-hand side, must be integrated numerically. To look for a similarly solution of this equation imposes an unrealistic distribution of vorticity behind the shock wave, as was pointed out by Hayes (1956).

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